Minimal Supersolutions of BSDEs with Lower Semicontinuous Generators*

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We study the existence and uniqueness of minimal supersolutions of backward stochastic differential equations with generators that are jointly lower semicontinuous, bounded below by an affine function of the control variable and satisfy a specific normalization property.

Keywords: Supersolutions of Backward Stochastic Differential Equations; Semimartingale Convergence; Nonlinear Expectations

1 Introduction

On a filtered probability space, the filtration of which is generated by a d-dimensional Brownian motion, we want to give conditions ensuring that the set $\mathcal{A}(\xi, g)$, consisting of all *supersolutions* (Y, Z) of a backward stochastic differential equation with *terminal condition* ξ and *generator* g, has a minimal element. Recall that such a supersolution is a pair (Y, Z) such that, for $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi$$

is satisfied. We call Y the *value process* and Z the *control process* of the supersolution (Y, Z). We start by considering the process $\mathcal{E}^g(\xi)$ defined as

$$\mathcal{E}^g_t(\xi) = \operatorname{ess\,inf}\left\{Y_t \in L^0(\mathcal{F}_t) \,:\, (Y, Z) \in \mathcal{A}(\xi, g)\right\}, \quad t \in [0, T],$$

and show that, under suitable conditions on the generator and the terminal condition, $\mathcal{E}^g(\xi)$ is already the value process of the unique minimal supersolution, that is, there is a unique control process $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. It was recently shown in Drapeau et al. [6] that, if the generator g is jointly lower semicontinuous in y and z, convex in z, monotone in y, and bounded below by an affine function of

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z, a unique minimal supersolution exists. Their proof was based on convex combinations of a monotone decreasing sequence of càdlàg supermartingales converging pointwise to $\mathcal{E}^g(\xi)$ on the rationals and made use of compactness results for sequences of martingales given in Delbaen and Schachermayer [3]. We follow a different approach to show the existence of a unique minimal supersolution. Starting with the assumption that g is also jointly lower semicontinuous in y and z and *positive* and in addition satisfies a certain *normalization* condition, we find a sequence of supersolutions converging *uniformly* to the càdlàg supermartingale $\mathcal{E}^{g,+}(\xi)$, the right limit of $\mathcal{E}^g(\xi)$. We then use results on convergence of semimartingales given in Barlow and Protter [1] to identify a unique process $\hat{Z}$ such that $(\mathcal{E}^{g,+}(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. By showing that $\mathcal{E}^{g,+}(\xi)$ always stays below $\mathcal{E}^g(\xi)$, we deduce $\mathcal{E}^{g,+}(\xi) = \mathcal{E}^g(\xi)$ and thus $(\mathcal{E}^g(\xi), \hat{Z})$ is the unique minimal supersolution. Later on, we relax the positivity assumption to that of boundedness below by an affine function of z. Also the normalization condition will be relaxed. Hence both, Drapeau et al. [6] and our work, show the existence of a unique minimal supersolution of BSDEs, but under mutually singular conditions on the generator.

Let us briefly discuss the existing literature on related problems, a broader discussion of which can be found in Drapeau et al. [6]. Nonlinear BSDEs were first introduced in Pardoux and Peng [11]. They gave existence and uniqueness results for the case of Lipschitz generators and square integrable terminal conditions. Kobylanski [10] studies BSDEs with quadratic generators, whereas Delbaen et al. [4] consider superquadratic BSDEs with positive generators that are convex in z and independent of y. Among the first introducing supersolutions of BSDEs were El Karoui et al. [7, Section 2.3]. Further references can also be found in Peng [13], who studies the existence and uniqueness of minimal supersolutions under the assumption of a Lipschitz generator and square integrable terminal conditions. Most recently, Cheridito and Stadje [2] have analyzed existence and stability of supersolutions of BSDEs. They consider terminal conditions which are functionals of the underlying Brownian motion and generators that are convex in z and Lipschitz in y and they work with discrete time approximations of BSDEs. Furthermore, the concept of supersolutions is closely related to Peng's g and G-expectations, see for instance [12, 14, 15], since the mapping $\xi \mapsto \mathcal{E}_0^g(\xi)$ can be seen as a nonlinear expectation. Another related field are stochastic target problems as introduced in Soner and Touzi [20], the solutions of which are obtained by dynamic programming methods and can be characterized as viscosity solutions of second order PDEs.

The remainder of this paper is organized as follows. Setting and notations are specified in Section 2. A precise definition of minimal supersolutions and important structural properties of $\mathcal{E}^g(\xi)$, along with the main existence theorem, can then be found in Sections 3.1 and 3.2. Finally, possible relaxations on the assumptions imposed on the generator, as well as a generalization to the case of arbitrary continuous local martingales, are discussed in Section 3.3.

2 Setting and Notations

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where the filtration $(\mathcal{F}_t)$ is generated by a d-dimensional Brownian motion W and is assumed to satisfy the usual conditions. For some fixed time horizon $T > 0$ and for all $t \in [0, T]$, the sets of $\mathcal{F}_t$-measurable random variables are denoted by $L^0(\mathcal{F}_t)$, where random variables are identified in the P-almost sure sense. Let furthermore denote $L^p(\mathcal{F}_t)$ the set of random variables in $L^0(\mathcal{F}_t)$ with finite p-norm, for $p \in [1, +\infty]$. Inequalities and strict inequalities between any two random variables or processes X^1, X^2 are understood in the P-almost sure or in the $P \otimes dt$-almost sure sense, respectively. In particular, two *càdlàg* processes X^1, X^2 satisfying $X^1 = X^2$

are indistinguishable, compare [8, Chapter III]. We denote by $\mathcal{T}$ the set of stopping times with values in $[0,T]$ and hereby call an increasing sequence of stopping times (τ^n), such that $P[\bigcup_n \{\tau^n = T\}] = 1$, a *localizing sequence of stopping times*. By $\mathcal{S} := \mathcal{S}(\mathbb{R})$ we denote the set of càdlàg progressively measurable processes Y with values in $\mathbb{R}$. For $p \in [1, +\infty[$, we further denote by $\mathcal{H}^p$ the set of càdlàg local martingales M with finite $\mathcal{H}^p$-norm on $[0,T]$, that is $\|M\|_{\mathcal{H}^p} := E[\langle M, M\rangle_T^{p/2}]^{1/p} < \infty$. By $\mathcal{L}^p := \mathcal{L}^p(W)$ we denote the set of $\mathbb{R}^{1\times d}$-valued, progressively measurable processes Z, such that $\int Z dW \in \mathcal{H}^p$, that is, $\|Z\|_{\mathcal{L}^p} := E[(\int_0^T Z_s^2 ds)^{p/2}]^{1/p}$ is finite. For $Z \in \mathcal{L}^p$, the stochastic integral $(\int_0^t Z_s dW_s)_{t\in[0,T]}$ is well defined, see [16], and is by means of the Burkholder-Davis-Gundy inequality [16, Theorem 48] a continuous martingale. We further denote by $\mathcal{L} := \mathcal{L}(W)$ the set of $\mathbb{R}^{1\times d}$-valued, progressively measurable processes Z, such that there exists a localizing sequence of stopping times (τ^n) with $Z1_{[0,\tau^n]} \in \mathcal{L}^1$, for all $n \in \mathbb{N}$. For $Z \in \mathcal{L}$, the stochastic integral $\int Z dW$ is well defined and is a continuous local martingale. Furthermore, for a process X, let X^* denote the following expression $X^* := \sup_{t\in[0,T]} |X_t|$, by which we define the norm $\|X\|_{\mathcal{R}^\infty} := \|X^*\|_{L^\infty}$.

We call a càdlàg semimartingale X a *special semimartingale*, if it can be decomposed into $X = X_0 + M + A$, where M is a local martingale and A a predictable process of finite variation such that $M_0 = A_0 = 0$. Such a decomposition is then unique, compare for instance [16, Chapter III, Theorem 30], and is called the *canonical* decomposition of X.

We will use *normal integrands*, a concept introduced in [18], as generators of BSDEs. Throughout this paper, a normal integrand is a function $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d} \to]-\infty, +\infty]$, such that

- $(y, z) \mapsto g(\omega, t, y, z)$ is jointly lower semicontinuous, for all $(\omega, t) \in \Omega \times [0,T]$;
- $(\omega, t) \mapsto g(\omega, t, y, z)$ is progressively measurable, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1\times d}$.

For a normal integrand g and progressively measurable processes Y, Z, the process $g(Y,Z)$ is itself progressively measurable and the integral $\int g(Y,Z)ds$ is well defined, P-almost surely, under the assumption that $+\infty - \infty = +\infty$, see [19, Chapter 14.F]. Finally, as long as $g \geq 0$, the lower semicontinuity yields an extended Fatou's lemma, that is,

$$\int \liminf_n g_s(Y_s^n, Z_s^n)\, ds \leq \liminf_n \int g_s(Y_s^n, Z_s^n)\, ds\,,$$

for any sequence $((Y^n, Z^n))$ of progressively measurable processes.

3 Minimal Supersolutions of BSDEs

3.1 First Definitions and Structural Properties

A pair $(Y,Z) \in \mathcal{S} \times \mathcal{L}$ is called a *supersolution* of a BSDE, if, for $0 \leq s \leq t \leq T$, it satisfies

$$Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi\,, \tag{3.1}$$

for a normal integrand g as *generator* and a *terminal condition* $\xi \in L^0(\mathcal{F}_T)$. For a supersolution (Y,Z), we call Y the *value process* and Z its corresponding *control process*. A control process Z is said to be

admissible, if the continuous local martingale $\int ZdW$ is a supermartingale.
Throughout this paper we say that a generator g is

(POS) positive, if $g(y,z) \geq 0$, for all $(y,z) \in \mathbb{R} \times \mathbb{R}^{1\times d}$.

(NOR) normalized, if $g_t(y,0) = 0$, for all $(t,y) \in [0,T] \times \mathbb{R}$.

We are now interested in the set

$$\mathcal{A}(\xi, g) := \{(Y,Z) \in \mathcal{S} \times \mathcal{L} \,:\, Z \text{ is admissible and (3.1) holds}\} \tag{3.2}$$

and the process

$$\mathcal{E}^g_t(\xi) = \operatorname{ess\,inf}\left\{Y_t \in L^0(\mathcal{F}_t) \,:\, (Y,Z) \in \mathcal{A}(\xi,g)\right\}\,, \quad t \in [0,T]\,. \tag{3.3}$$

For the proof of our main existence theorem we will need some auxiliary results concerning structural properties of $\mathcal{E}^g(\xi)$ and supersolutions (Y,Z) in $\mathcal{A}(\xi,g)$.

Lemma 3.1. *Let g be a generator satisfying* (POS)*. Assume further that $\mathcal{A}(\xi,g) \neq \emptyset$ and $\xi^- \in L^1(\mathcal{F}_T)$. Then $\xi \in L^1(\mathcal{F}_T)$ and, for any $(Y,Z) \in \mathcal{A}(\xi,g)$, the control Z is unique and the value process Y is a supermartingale such that $Y_t \geq E\left[\xi \,|\, \mathcal{F}_t\right]$. Moreover, the unique canonical decomposition of Y is given by*

$$Y = Y_0 + M - A\,, \tag{3.4}$$

where $M = \int ZdW$ and A is an increasing, predictable, càdlàg process with $A_0 = 0$.

The proof of Lemma 3.1 can be found in [6, Lemma 3.4].

Proposition 3.2. *Suppose that $\mathcal{A}(\xi,g) \neq \emptyset$ and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If g satisfies* (POS)*, then the process $\mathcal{E}^g(\xi)$ is a supermartingale and*

$$\mathcal{E}^g_t(\xi) \geq \mathcal{E}^{g,+}_t(\xi) := \lim_{\substack{s\downarrow t\\ s\in\mathbb{Q}}} \mathcal{E}^g_s(\xi)\,, \quad \textit{for all } t \in [0,T)\,, \tag{3.5}$$

and $\mathcal{E}^{g,+}_T(\xi) := \xi$. In particular, $\mathcal{E}^{g,+}(\xi)$ is a càdlàg supermartingale. Furthermore, the following two pasting properties hold true.

1. *Let $(Z^n) \subset \mathcal{L}$ be admissible, $\sigma \in \mathcal{T}$, and $(B_n) \subset \mathcal{F}_\sigma$ a partition of Ω. Then the pasted process $\bar{Z} = Z^1 1_{[0,\sigma]} + \sum_{n\geq 1} Z^n 1_{B_n} 1_{]\sigma,T]}$ is admissible.*
2. *Let $((Y^n,Z^n)) \subset \mathcal{A}(\xi,g)$, $\sigma \in \mathcal{T}$ and $(B_n) \subset \mathcal{F}_\sigma$ as before. If $Y^1_{\sigma-} 1_{B_n} \geq Y^n_\sigma 1_{B_n}$ holds true for all $n \in \mathbb{N}$, then $(\bar{Y},\bar{Z}) \in \mathcal{A}(\xi,g)$, where*

$$\bar{Y} = Y^1 1_{[0,\sigma[} + \sum_{n\geq 1} Y^n 1_{B_n} 1_{[\sigma,T]} \quad \textit{and} \quad \bar{Z} = Z^1 1_{[0,\sigma]} + \sum_{n\geq 1} Z^n 1_{B_n} 1_{]\sigma,T]}\,.$$

Proof. The proof of the part concerning the process $\mathcal{E}^{g,+}(\xi)$ can be found in [6, Proposition 3.5]. As to the first pasting property, let M^n and $\bar{M}$ denote the stochastic integrals of the Z^n and $\bar{Z}$, respectively. First, it follows from $(Z^n) \subset \mathcal{L}$ and from (B_n) being a partition that $\bar{Z} \in \mathcal{L}$ and that $\int_{s\vee\sigma}^{t\vee\sigma} \bar{Z}_u dW_u = \sum 1_{B_n} \int_{s\vee\sigma}^{t\vee\sigma} Z^n_u dW_u$. Now observe that the admissibility of all Z^n yields

$$E\left[\bar{M}_t - \bar{M}_s \,|\, \mathcal{F}_s\right] = E\left[M^1_{(t\wedge\sigma)\vee s} - M^1_s \,|\, \mathcal{F}_s\right] + E\Big[\sum_{n\geq 1} 1_{B_n} E\left[M^n_{t\vee\sigma} - M^n_{s\vee\sigma} \,|\, \mathcal{F}_{s\vee\sigma}\right] \,\Big|\, \mathcal{F}_s\Big] \leq 0\,,$$

for $0 \le s \le t \le T$. Thus, $\bar{Z}$ is admissible. Finally, we can approximate σ from below by some foretelling sequence of stopping times (η_m)[1], and then show, analogously to [6, Lemma 3.1], that the pair $(\bar{Y}, \bar{Z})$ satisfies inequality (3.1) and is thus an element of $\mathcal{A}(\xi, g)$. $\square$

Proposition 3.3. *Let $0 = \tau_0 \le \tau_1 \le \tau_2 \le \cdots$ be a sequence of stopping times converging to the finite stopping time $\tau^* = \lim_{n\to\infty} \tau_n$. Further, let (Y^n) be a sequence of càdlàg supermartingales such that $Y^n_{\tau_n -} \ge Y^{n+1}_{\tau_n}$, and which satisfies $Y^n 1_{[\tau_{n-1},\tau_n[} \ge M 1_{[\tau_{n-1},\tau_n[}$, where M is a uniformly integrable martingale. Then, for any sequence of stopping times $\sigma_n \in [\tau_{n-1}, \tau_n[$, the limit $Y^\infty := \lim_{n\to\infty} Y^n_{\sigma_n}$ exists and the process*

$$\bar{Y} := \sum_{n\ge 1} Y^n 1_{[\tau_{n-1},\tau_n[} + Y^\infty 1_{[\tau^*,\infty[}$$

is a càdlàg supermartingale. Moreover, the limit Y^∞ is independent of the approximating sequence $(Y^n_{\sigma_n})$ and, if all Y^n are continuous and $Y^n_{\tau_n} = Y^{n+1}_{\tau_n}$, for all $n \in \mathbb{N}$, then $\bar{Y}$ is continuous.

Proof. Note that $(Y^n_{\sigma_n})$ is a $(\mathcal{F}_{\sigma_n})$-supermartingale. Indeed, if $(\tilde{\eta}_m) \uparrow \tau_n$ is a foretelling sequence of stopping times, then, with $\eta_m := \tilde{\eta}_m \vee \tau_{n-1}$, the family $((Y^n_{\eta_m})^-)_{m\in\mathbb{N}}$ is uniformly integrable and we obtain

$$\begin{aligned} E[Y^{n+1}_{\sigma_{n+1}} \,|\, \mathcal{F}_{\sigma_n}] = E[E[Y^{n+1}_{\sigma_{n+1}} \,|\, \mathcal{F}_{\tau_n}] \,|\, \mathcal{F}_{\sigma_n}] \le E[Y^{n+1}_{\tau_n} \,|\, \mathcal{F}_{\sigma_n}] \le E[Y^n_{\tau_n -} \,|\, \mathcal{F}_{\sigma_n}] \\ \le \liminf_m E[Y^n_{\eta_m} \,|\, \mathcal{F}_{\sigma_n}] \le \liminf_m Y^n_{\eta_m \wedge \sigma_n} = Y^n_{\sigma_n} \,. \end{aligned}$$

Moreover, $((Y^n_{\sigma_n})^-)$ is uniformly integrable. Hence, the sequence $(Y^n_{\sigma_n})$ converges by the supermartingale convergence theorem, see [5, Theorems V.28,29], to some random variable Y^∞, P-almost surely, and thus $\bar{Y}$ is well-defined. Furthermore, the limit Y^∞ is independent of the approximating sequence $(Y^n_{\sigma_n})$. Indeed, for any other sequence $(\tilde{\sigma}_n)$ with $\tilde{\sigma}_n \in [\tau_{n-1}, \tau_n[$, the limit $\lim_n Y^n_{\tilde{\sigma}_n}$ exists by the same argumentation. Now $\lim_n Y^n_{\sigma_n} = \lim_n Y^n_{\tilde{\sigma}_n} = Y^\infty$ holds, since the sequence $(\hat{\sigma}_n)$ defined by

$$\hat{\sigma}_n := \begin{cases} \sigma_{\frac{n}{2}} \vee \tilde{\sigma}_{\frac{n}{2}} & \text{, for } n \text{ even} \\ \sigma_{\frac{n+1}{2}} \wedge \tilde{\sigma}_{\frac{n+1}{2}} & \text{, for } n \text{ odd} \end{cases}$$

satisfies $\hat{\sigma}_n \in [\tau_{n-1}, \tau_n[$ and $\lim_n Y^n_{\hat{\sigma}_n}$ exists. Thus, all limits must coincide. Next, we show that $\bar{Y}^{\sigma_n}$ is a supermartingale, for all $n \in \mathbb{N}$. To this end first observe that, for all $0 \le s \le t$,

$$\begin{aligned} E\left[\bar{Y}^{\sigma_n}_t - \bar{Y}^{\sigma_n}_s \,|\, \mathcal{F}_s\right] = \sum_{k=0}^{n-2} E\big[E\big[\bar{Y}^{\sigma_n}_{(\tau_{k+1}\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k \vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] \,|\, \mathcal{F}_s\big] \\ + E\big[E\big[\bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_{n-1}\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_{n-1}\vee s)\wedge t}\big] \,|\, \mathcal{F}_s\big] \\ + E\big[E\big[\bar{Y}^{\sigma_n}_t - \bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} \,|\, \mathcal{F}_{(\sigma_n\vee s)\wedge t}\big] \,|\, \mathcal{F}_s\big] \,. \end{aligned}$$

Note further that, for each $n \in \mathbb{N}$, the process $\bar{Y}^{\sigma_n}$ is càdlàg and can only jump downwards, that is, $\bar{Y}^{\sigma_n}_{t-} \ge \bar{Y}^{\sigma_n}_t$, for all $t \in \mathbb{R}$. Observe to this end that, on the one hand, $\bar{Y}^{\sigma_n}_{\tau_k -} = Y^k_{\tau_k -} \ge Y^{k+1}_{\tau_k} = \bar{Y}^{\sigma_n}_{\tau_k}$, for all $0 \le k \le n-1$, by assumption, where we assumed $\tau_{k-1} < \tau_k$, without loss of generality. On the other hand, Y^k can only jump downwards. Indeed, as càdlàg supermartingales, all Y^k can be decomposed into $Y^k = Y^k_0 + M^k - A^k$, by the Doob-Meyer decomposition theorem [16, Chapter III, Theorem 13], where

[1] Such a sequence satisfying $\tilde{\eta}_m < \sigma$, for all $m \in \mathbb{N}$, always exists, since in a Brownian filtration every stopping time is predictable, compare [17, Corollary V.3.3].

M^k is a local martingale and A^k a predictable, increasing process with $A_0^k = 0$. Since in a Brownian filtration every local martingale is continuous, the claim follows.
Thus, for all $0 \leq k \leq n-2$, and $(\tilde{\eta}_m) \uparrow \tau_{k+1}$ a foretelling sequence of stopping times, it holds with $\eta_m := \tilde{\eta}_m \vee \tau_k$,

$$
\begin{aligned}
E\big[\bar{Y}^{\sigma_n}_{(\tau_{k+1}\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] &\leq E\big[\bar{Y}^{\sigma_n}_{((\tau_{k+1}-)\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] \\
&= E\big[\liminf_m \bar{Y}^{\sigma_n}_{(\eta_m\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] \\
&\leq E\big[\liminf_m Y^{k+1}_{(\eta_m\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] - Y^{k+1}_{(\tau_k\vee s)\wedge t} \\
&\leq \liminf_m E\big[Y^{k+1}_{(\eta_m\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] - Y^{k+1}_{(\tau_k\vee s)\wedge t} \leq 0\,.
\end{aligned}
$$

Moreover, $E[\bar{Y}^{\sigma_n}_t - \bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} \,|\, \mathcal{F}_{(\sigma_n\vee s)\wedge t}] = 0$, as well as

$$
E\big[\bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_{n-1}\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_{n-1}\vee s)\wedge t}\big] \;\leq\; E\big[Y^n_{(\sigma_n\vee s)\wedge t} - Y^n_{(\tau_{n-1}\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_{n-1}\vee s)\wedge t}\big] \;\leq\; 0\,.
$$

Combining this we obtain that $E\left[\bar{Y}^{\sigma_n}_t \,|\, \mathcal{F}_s\right] \leq \bar{Y}^{\sigma_n}_s$. Furthermore, $\lim_n \bar{Y}^{\sigma_n}_t = \bar{Y}_t$, for all $t \in \mathbb{R}$. Indeed, let us write $\lim_n \bar{Y}^{\sigma_n}_t = \lim_n \bar{Y}^{\sigma_n}_t 1_{\{t<\tau^*\}} + \lim_n \bar{Y}^{\sigma_n}_t 1_{\{t\geq\tau^*\}}$. Then, $\lim_n \bar{Y}^{\sigma_n}_t 1_{\{t\geq\tau^*\}} = \lim_n Y^n_{\sigma_n} 1_{\{t\geq\tau^*\}} = Y^\infty 1_{\{t\geq\tau^*\}} = \bar{Y}_t 1_{\{t\geq\tau^*\}}$ and $\lim_n \bar{Y}_{\sigma_n\wedge t} 1_{\{t<\tau^*\}} = \bar{Y}_t 1_{\{t<\tau^*\}}$. Hence, the claim follows. As a consequence of Fatou's lemma it now holds that

$$
E\left[\bar{Y}_t \,|\, \mathcal{F}_s\right] \leq \liminf_{n\to\infty} E\left[\bar{Y}^{\sigma_n}_t \,|\, \mathcal{F}_s\right] \leq \liminf_{n\to\infty} \bar{Y}^{\sigma_n}_s = \bar{Y}_s\,,
$$

since the family $((\bar{Y}^{\sigma_n}_t)^-)$ is uniformly integrable. Hence, $\bar{Y}$ is a supermartingale, which by construction has right-continuous paths and [9, Theorem 1.3.8] then yields that $\bar{Y}$ is even càdlàg. Finally, whenever all Y^n are continuous and $Y^n_{\tau_n} = Y^{n+1}_{\tau_n}$ holds, for all $n \in \mathbb{N}$, the process $\bar{Y}$ is continuous per construction.□

3.2 Existence and Uniqueness of Minimal Supersolutions

We are now ready to state our main existence result. Possible relaxations of the assumptions (POS) and (NOR) imposed on the generator are discussed in Section 3.3. Note that it is not our focus to investigate conditions assuring the crucial assumption that $\mathcal{A}(\xi, g) \neq \emptyset$. See Drapeau et al. [6] and the references therein for further details.

Theorem 3.4. *Let g be a generator satisfying* (POS) *and* (NOR) *and $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then there exists a unique $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Proof. Step 1: Uniform Limit and Verification. Since $\mathcal{A}(\xi, g) \neq \emptyset$, there exist $(Y^b, Z^b) \in \mathcal{A}(\xi, g)$. From now on we restrict our focus to supersolutions $(\bar{Y}, \bar{Z})$ in $\mathcal{A}(\xi, g)$ satisfying $\bar{Y}_0 \leq Y_0^b$. Indeed, since we are only interested in minimal supersolutions, we can paste any value process of $(Y, Z) \in \mathcal{A}(\xi, g)$ at $\tau := \inf\{t > 0 \,:\, Y^b_t > Y_t\} \wedge T$, such that $\bar{Y} := Y^b 1_{[0,\tau[} + Y 1_{[\tau,T]}$ satisfies $\bar{Y}_0 \leq Y_0^b$, where the corresponding control $\bar{Z}$ is obtained as in Proposition 3.2.

Assume for the beginning that we can find a sequence $((Y^n, Z^n))$ within $\mathcal{A}(\xi, g)$ such that

$$
\lim_{n\to\infty} \left\|Y^n - \mathcal{E}^{g,+}(\xi)\right\|_{\mathcal{R}^\infty} = 0\,. \tag{3.6}
$$

Since all Y^n are càdlàg supermartingales, they are, by the Doob-Meyer decomposition theorem, see [16, Chapter III, Theorem 13], special semimartingales with canonical decomposition $Y^n = Y^n_0 + M^n - A^n$ as

in (3.4). The supermartingale property of all $\int Z^n dW$ and $\xi \in L^1(\mathcal{F}_T)$, compare Lemma 3.1, imply that $E\left[A^n_T\right] \leq Y^b_0 - E\left[\xi\right] \in L^1(\mathcal{F}_T)$. Hence, since each A^n is increasing, $\sup_n E[\int_0^T |dA^n_s|] < \infty$. As (3.6) implies in particular that $\lim_{n\to\infty} E\left[(Y^n - \mathcal{E}^{g,+}(\xi))^*\right] = 0$, it follows from [1, Theorem 1 and Corollary 2] that $\mathcal{E}^{g,+}(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^{g,+}(\xi) = \mathcal{E}^{g,+}_0(\xi) + M - A$ and that

$$\lim_{n\to\infty} \|M^n - M\|_{\mathcal{H}^1} = 0 \quad , \quad \lim_{n\to\infty} E\left[(A^n - A)^*\right] = 0\,.$$

The local martingale M is continuous and allows for a representation of the form $M = M_0 + \int \hat{Z} dW$, where $\hat{Z} \in \mathcal{L}$, compare [16, Chapter IV, Theorem 43]. Since

$$E\left[\left(\int\limits_0^T \left(Z^n_u - \hat{Z}_u\right)^2 du\right)^{1/2}\right] \xrightarrow[n\to+\infty]{} 0\,,$$

we have that, up to a subsequence, (Z^n) converges $P \otimes dt$-almost surely to $\hat{Z}$ and $\lim_{n\to\infty} \int_0^t Z^n dW = \int_0^t \hat{Z} dW$, for all $t \in [0,T]$, P-almost surely, due to the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n\to\infty} Z^n(\omega) = \hat{Z}(\omega)$, dt-almost surely, for almost all $\omega \in \Omega$.
In order to verify that $(\mathcal{E}^{g,+}(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$, we will use the convergence obtained above. More precisely, for all $0 \leq s \leq t \leq T$, Fatou's lemma together with (3.6) and the lower semicontinuity of the generator yields

$$\begin{aligned}
&\mathcal{E}^{g,+}_s(\xi) - \int\limits_s^t g_u(\mathcal{E}^{g,+}_u(\xi), \hat{Z}_u) du + \int\limits_s^t \hat{Z}_u dW_u \\
&\qquad\qquad \geq \limsup_n \left(Y^n_s - \int\limits_s^t g_u(Y^n_u, Z^n_u) du + \int\limits_s^t Z^n_u dW_u\right) \geq \limsup_n Y^n_t = \mathcal{E}^{g,+}_t(\xi)\,.
\end{aligned}$$

The above, the positivity of g and $\mathcal{E}^{g,+}(\xi) \geq E\left[\xi \,|\, \mathcal{F}_\cdot\right]$ imply that $\int \hat{Z} dW \geq E\left[\xi \,|\, \mathcal{F}_\cdot\right] - \mathcal{E}^{g,+}_0(\xi)$. Hence, being bounded from below by a martingale, the continuous local martingale $\int \hat{Z} dW$ is a supermartingale. Thus, $\hat{Z}$ is admissible and $(\mathcal{E}^{g,+}(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ and therefore, by Lemma 3.1, $\hat{Z}$ is unique. Since we know by Proposition 3.2 that $\mathcal{E}^g(\xi) \geq \mathcal{E}^{g,+}(\xi)$, we deduce that $\mathcal{E}^g(\xi) = \mathcal{E}^{g,+}(\xi)$ by the definition of $\mathcal{E}^g(\xi)$, identifying $(\mathcal{E}^g(\xi), \hat{Z})$ as the unique minimal supersolution.

Step 2: A preorder on $\mathcal{A}(\xi, g)$. As to the existence of $((Y^n, Z^n))$ satisfying (3.6), it is sufficient to show that, for arbitrary $\varepsilon > 0$, we can find a supersolution $(Y^\varepsilon, Z^\varepsilon)$ satisfying

$$\left\|Y^\varepsilon - \mathcal{E}^{g,+}(\xi)\right\|_{\mathcal{R}^\infty} \leq \varepsilon\,. \tag{3.7}$$

We define the following preorder[2] on $\mathcal{A}(\xi, g)$

$$(Y^1, Z^1) \preceq (Y^2, Z^2) \;\;\Leftrightarrow\;\; \tau_1 \leq \tau_2 \;\text{ and }\; (Y^1, Z^1) 1_{[0,\tau_1[} = (Y^2, Z^2) 1_{[0,\tau_1[}\,, \tag{3.8}$$

where, for $i = 1, 2$,

$$\tau_i = \inf\left\{t \geq 0 \;:\; Y^i_t > \mathcal{E}^{g,+}_t(\xi) + \varepsilon\right\} \wedge T\,. \tag{3.9}$$

[2] Note that, in order to apply Zorn's lemma, we need a partial order instead of just a preorder. To this end we consider equivalence classes of processes. Two supersolutions $(Y^1, Z^1), (Y^2, Z^2) \in \mathcal{A}(\xi, g)$ are said to be equivalent, that is, $(Y^1, Z^1) \sim (Y^2, Z^2)$, if $(Y^1, Z^1) \preceq (Y^2, Z^2)$ and $(Y^2, Z^2) \preceq (Y^1, Z^1)$. This means that they are equal up to their corresponding stopping time $\tau_1 = \tau_2$ as in (3.9). This induces a partial order on the set of equivalence classes and hence the use of Zorn's lemma is justified.

For any totally ordered chain $((Y^i, Z^i))_{i\in I}$ within $\mathcal{A}(\xi, g)$ with corresponding stopping times τ_i, we want to construct an upper bound. If we consider

$$\tau^* = \operatorname*{ess\,sup}_{i\in I} \tau_i \,,$$

we know by the monotonicity of the stopping times that we can find a monotone subsequence (τ_m) of $(\tau_i)_{i\in I}$ such that $\tau^* = \lim_{m\to\infty} \tau_m$. In particular, τ^* is a stopping time. Furthermore, the structure of the preorder (3.8) yields that the value processes of the supersolutions $((Y^m, Z^m))$ corresponding to the stopping times (τ_m) satisfy

$$Y^{m+1}_{\tau_m} \le Y^{m+1}_{\tau_m -} = Y^m_{\tau_m -} \quad , \quad \text{for all } m \in \mathbb{N}\,, \tag{3.10}$$

where the inequality follows from the fact that all Y^m are càdlàg supermartingales, see the proof of Proposition 3.3.

Step 3: A candidate upper bound $(\bar{Y}, \bar{Z})$ *for the chain* $((Y^i, Z^i))_{i\in I}$. We construct a candidate upper bound $(\bar{Y}, \bar{Z})$ for $((Y^i, Z^i))_{i\in I}$ satisfying $P[\tau(\bar{Y}) > \tau^* \,|\, \tau^* < T] = 1$, with $\tau(\bar{Y})$ as in (3.9).
To this end, let $(\bar{\sigma}_n)$ be a decreasing sequence of stopping times taking values in the rationals and converging towards τ^* from the right[3]. Then the stopping times $\hat{\sigma}_n := \bar{\sigma}_n \wedge T$ satisfy $\hat{\sigma}_n > \tau^*$ and $\hat{\sigma}_n \in \mathbb{Q}$, on $\{\tau^* < T\}$, for all n big enough. Let us furthermore define the following stopping time

$$\bar{\tau} := \inf\left\{ t > \tau^* \;:\; 1_{\{\tau^* < T\}} \left| \mathcal{E}^{g,+}_{\tau^*}(\xi) - \mathcal{E}^{g,+}_{t}(\xi) \right| > \frac{\varepsilon}{2} \right\} \wedge T\,. \tag{3.11}$$

Due to the right-continuity of $\mathcal{E}^{g,+}(\xi)$ in τ^*, it follows that $\bar{\tau} > \tau^*$ on $\{\tau^* < T\}$. We now set

$$\sigma_n := \hat{\sigma}_n \wedge \bar{\tau}\,, \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

The above stopping times still satisfy $\lim_{n\to\infty} \sigma_n = \tau^*$ and $\sigma_n > \tau^*$ on $\{\tau^* < T\}$, for all $n \in \mathbb{N}$. We further define the following sets

$$A_n := \left\{ \left| \mathcal{E}^{g,+}_{\tau^*}(\xi) - \mathcal{E}^{g}_{\sigma_m}(\xi) \right| \vee \left| \mathcal{E}^{g,+}_{\tau^*}(\xi) - \mathcal{E}^{g,+}_{\sigma_m}(\xi) \right| < \frac{\varepsilon}{8}\,, \quad \text{for all } m \ge n \right\}\,. \tag{3.13}$$

They satisfy $A_n \subset A_{n+1}$ and $\bigcup_n A_n = \Omega$ by definition of the sequence (σ_m)[4]. Note further that $A_n \in \mathcal{F}_{\sigma_n}$ holds true by construction. By Proposition 3.2 we deduce[5] that, for each $n \in \mathbb{N}$, there exist $(\tilde{Y}^n, \tilde{Z}^n) \in \mathcal{A}(\xi, g)$ such that

$$\tilde{Y}^n_{\sigma_n} \le \mathcal{E}^g_{\sigma_n}(\xi) + \frac{\varepsilon}{8}\,. \tag{3.14}$$

Next we partition Ω into $B_n := A_n \backslash A_{n-1}$, where we set $A_0 := \emptyset$ and $\tau_0 := 0$, and define the candidate upper bound as

$$\bar{Y} = \sum_{m\ge1} Y^m 1_{[\tau_{m-1}, \tau_m[} + 1_{\{\tau^* < T\}} \sum_{n\ge1} 1_{B_n} \left[\left(\mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{\varepsilon}{2} \right) 1_{[\tau^*, \sigma_n[} + \tilde{Y}^n 1_{[\sigma_n, T[} \right] \quad , \quad \bar{Y}_T = \xi\,, \tag{3.15}$$

$$\bar{Z} = \sum_{m\ge1} Z^m 1_{]\tau_{m-1}, \tau_m]} + 1_{\{\tau^* < T\}} \sum_{n\ge1} \tilde{Z}^n 1_{B_n} 1_{]\sigma_n, T]}\,. \tag{3.16}$$

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[3] Compare [9, Problem 2.24].
[4] Since on $\{\tau^* < T\}$, $\bar{\tau} > \tau^*$ and $\lim_n \hat{\sigma}_n = \tau^*$ with $\hat{\sigma}_n \in \mathbb{Q}$, it is ensured that there exists some $n_0 \in \mathbb{N}$, depending on ω, such that σ_n takes values in the rationals for all $n \ge n_0$. By definition of $\mathcal{E}^{g,+}(\xi)$ as the right-hand side limit of $\mathcal{E}^g(\xi)$ on the rationals and due to the right-continuity of $\mathcal{E}^{g,+}(\xi)$ in τ^*, both inequalities in the definition of A_n are satisfied for all $n \ge n_0$.
[5] For a detailed proof, see [6, Proposition 3.2.].

Step 4: Verification of $(\bar{Y},\bar{Z})\in\mathcal{A}(\xi,g)$. By verifying that the pair $(\bar{Y},\bar{Z})$ is an element of $\mathcal{A}(\xi,g)$, we identify $(\bar{Y},\bar{Z})$ as an upper bound for the chain $((Y^i,Z^i))_{i\in I}$. Even more, $P[\tau(\bar{Y})>\tau^*\,|\,\tau^*<T]=1$ holds true, since, on the set B_n, we have $\bar{Y}_t=\mathcal{E}^{g,+}_{\tau^*}(\xi)+\frac{\varepsilon}{2}\leq\mathcal{E}^{g,+}_t(\xi)+\varepsilon$, for all $t\in[\tau^*,\sigma_n[$, due to the definition of $\bar{\tau}$ in (3.11).

Step 4a: The value process $\bar{Y}$ *is an element of* $\mathcal{S}$. By construction, the only thing to show is that $\bar{Y}_{\tau^*-}$, the left limit at τ^*, exists. This follows from Proposition 3.3, since, by means of $((Y^m,Z^m))\subset\mathcal{A}(\xi,g)$ and $\xi\in L^1(\mathcal{F}_T)$, all Y^m are càdlàg supermartingales, see Lemma 3.1, which are bounded from below by a uniformly integrable martingale, more precisely $Y^m\geq E\left[\xi\,|\,\mathcal{F}_\cdot\right]$, for all $m\in\mathbb{N}$, and satisfy (3.10).

Step 4b: The control process $\bar{Z}$ *is an element of* $\mathcal{L}$ *and admissible.* We proceed by defining, for each $n\in\mathbb{N}$, the processes $\bar{Z}^n:=\sum_{m=1}^n Z^m 1_{]\tau_{m-1},\tau_m]}=\bar{Z}1_{[0,\tau_n]}=Z^n 1_{[0,\tau_n]}$ and $N^n:=\int\bar{Z}^n dW=\int Z^n 1_{[0,\tau_n]}dW$, where the equalities follow from (3.8). Observe that $N^{n+1}1_{[0,\tau_n]}=N^n 1_{[0,\tau_n]}$, for all $n\in\mathbb{N}$, and that (POS), (3.1) and the supermartingale property of $\int Z^n dW$ imply

$$N^n 1_{[\tau_{n-1},\tau_n[}\geq 1_{[\tau_{n-1},\tau_n[}(-E\left[\xi^-\,|\,\mathcal{F}_\cdot\right]-Y^b_0)\,. \tag{3.17}$$

By means of (3.17) and since $\xi^-\in L^1(\mathcal{F}_T)$, with $N^\infty:=\lim_n N^n_{\tau_{n-1}}$, the process

$$N=\sum_{n\geq 1}N^n 1_{[\tau_{n-1},\tau_n[}+1_{[\tau^*,T]}N^\infty$$

is a well-defined continuous supermartingale due to Proposition 3.3. Hence we may define a localizing sequence by setting $\kappa_n:=\inf\{t\geq 0\,:\,|N_t|>n\}\wedge T$ and deduce that N is a continuous local martingale, because N^{κ_n} is a uniformly integrable martingale, for all $n\in\mathbb{N}$. Indeed, for each $n\in\mathbb{N}$ and $m\in\mathbb{N}$, the process $(N^m)^{\kappa_n}$, being a bounded stochastic integral, is a martingale. Moreover, the family $(N^m_{\kappa_n\wedge t})_{m\in\mathbb{N}}$ is uniformly integrable and $N_{\kappa_n\wedge t}=\lim_m N^m_{\kappa_n\wedge t}$, for all $t\in[0,T]$. Consequently, $E[N^{\kappa_n}_t\,|\,\mathcal{F}_s]=\lim_m E[N^m_{\kappa_n\wedge t}\,|\,\mathcal{F}_s]=\lim_m N^m_{\kappa_n\wedge s}=N^{\kappa_n}_s$, for all $0\leq s\leq t\leq T$, and the claim follows. Since the quadratic variation of a continuous local martingale is continuous and unique, see [9, page 36], we obtain

$$\int_0^{\tau^*}\bar{Z}^2_u du=\lim_n\int_0^{\kappa_n\wedge\tau^*}\bar{Z}^2_u du=\lim_n\langle N\rangle_{\kappa_n\wedge\tau^*}=\langle N\rangle_{\tau^*}<\infty\,.$$

Observe that $\sigma:=\sum_{n\geq 1}1_{B_n}\sigma_n$ is an element of $\mathcal{T}$. Indeed, $\{\sigma\leq t\}=\bigcup_{n\geq 1}(B_n\cap\{\sigma\leq t\})=\bigcup_{n\geq 1}(B_n\cap\{\sigma_n\leq t\})\in\mathcal{F}_t$, for all $t\in[0,T]$, since $B_n\in\mathcal{F}_{\sigma_n}$. From $\bar{Z}1_{]\tau^*,\sigma]}=0$ we get that

$$\int_0^T\bar{Z}^2_u du=\langle N\rangle_{\tau^*}+1_{\{\tau^*<T\}}\sum_{n\geq 1}1_{B_n}\int_\sigma^T(\tilde{Z}^n_u)^2du<\infty\,,$$

since $(\tilde{Z}^n)\subset\mathcal{L}$. Hence we conclude that $\bar{Z}\in\mathcal{L}$. As for the supermartingale property of $\int\bar{Z}dW$, observe that

$$\int_0^{t\wedge\tau^*}\bar{Z}_u dW_u=\lim_{n\to\infty}\int_0^{t\wedge\tau_n}Z^n_u dW_u\geq\lim_{n\to\infty}-E\left[\xi^-\,|\,\mathcal{F}_{t\wedge\tau_n}\right]-Y^b_0=-E\left[\xi^-\,|\,\mathcal{F}_{t\wedge\tau^*}\right]-Y^b_0\,,$$

where the inequality follows from (3.1) and (POS). Being bounded from below by a martingale, we deduce by Fatou's lemma that $\bar{Z}1_{[0,\tau^*]}$ is admissible. Since $\bar{Z}1_{]\tau^*,\sigma]} = 0$ and all $\tilde{Z}^n$ are admissible, it follows from Proposition 3.2 that $\bar{Z}$ is indeed admissible.

Step 4c: The pair $(\bar{Y}, \bar{Z})$ is a supersolution. Finally, showing that $(\bar{Y}, \bar{Z})$ satisfy (3.1) identifies $(\bar{Y}, \bar{Z})$ as an element of $\mathcal{A}(\xi, g)$. Observe first that, for all $0 \leq s \leq t \leq T$ and all $m \in \mathbb{N}$, the expression $\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u$ can be written as

$$\begin{aligned} \bar{Y}_s - \int_s^{(\tau_m \vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^{(\tau_m \vee s)\wedge t} \bar{Z}_u dW_u \\ - \int_{(\tau_m \vee s)\wedge t}^{(\tau^* \vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\tau_m \vee s)\wedge t}^{(\tau^* \vee s)\wedge t} \bar{Z}_u dW_u - \int_{(\tau^* \vee s)\wedge t}^{(\sigma \vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du \\ + \int_{(\tau^* \vee s)\wedge t}^{(\sigma \vee s)\wedge t} \bar{Z}_u dW_u - \int_{(\sigma \vee s)\wedge t}^{t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma \vee s)\wedge t}^{t} \bar{Z}_u dW_u \, . \end{aligned} \tag{3.18}$$

Now, we have that

$$\bar{Y}_s - \int_s^{(\tau_m \vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^{(\tau_m \vee s)\wedge t} \bar{Z}_u dW_u \;\geq\; \bar{Y}_{(\tau_m \vee s)\wedge t} \, , \tag{3.19}$$

by Proposition 3.2, since $((Y^m, Z^m)) \subset \mathcal{A}(\xi, g)$ and $Y^m_{\tau_m -} \geq Y^{m+1}_{\tau_m}$, for all $m \in \mathbb{N}$, due to (3.10). By letting m tend to infinity and noting that

$$\lim_{m\to\infty} \int_{(\tau_m \vee s)\wedge t}^{(\tau^* \vee s)\wedge t} \bar{Z}_u dW_u = 0 \quad \text{and} \quad \lim_{m\to\infty} \int_{(\tau_m \vee s)\wedge t}^{(\tau^* \vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du = 0 \, ,$$

(3.18) and (3.19) yield that

$$\begin{aligned} \bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u \\ \geq \bar{Y}_{((\tau^*-)\vee s)\wedge t} - \int_{(\tau^* \vee s)\wedge t}^{(\sigma \vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\tau^* \vee s)\wedge t}^{(\sigma \vee s)\wedge t} \bar{Z}_u dW_u \\ - \int_{(\sigma \vee s)\wedge t}^{t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma \vee s)\wedge t}^{t} \bar{Z}_u dW_u \, . \end{aligned} \tag{3.20}$$

We now use that $\bar{Y}$ can only jump downwards at τ^*. Indeed, since $\bar{Y}$ is càdlàg, in particular $\bar{Y}_{\tau^* -}$, the left limit at τ^*, exists and is unique, P-almost surely. Furthermore, $\lim_{m\to\infty} \bar{Y}_{\tau_m -} = \bar{Y}_{\tau^* -}$ and thus

$$\bar{Y}_{\tau^* -} = \lim_m \bar{Y}_{\tau_m -} = \lim_m Y^m_{\tau_m -} \geq \lim_m \mathcal{E}^{g,+}_{\tau_m}(\xi) + \varepsilon = \mathcal{E}^{g,+}_{\tau^* -}(\xi) + \varepsilon \geq \mathcal{E}^{g,+}_{\tau^*}(\xi) + \varepsilon > \bar{Y}_{\tau^*} \, .$$

The second inequality holds, since the càdlàg supermartingale $\mathcal{E}^{g,+}(\xi)$ can only jump downwards, see the proof of Proposition 3.3. Hence, (3.20) can be further estimated by

$$\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u \geq \bar{Y}_{(\tau^* \vee s)\wedge t} - \int_{(\sigma \vee s)\wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma\vee s)\wedge t}^t \bar{Z}_u dW_u \,, \quad (3.21)$$

where we used that

$$\int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du = \int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} \bar{Z}_u dW_u = 0\,,$$

due to (3.16), the definition of σ and (NOR). Now observe that $\bar{Y}_{(\tau^*\vee s)\wedge t} \geq \bar{Y}_{(\sigma\vee s)\wedge t}$, since $\bar{Y}1_{[\tau^*,\sigma[} = (\mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{\varepsilon}{2})1_{[\tau^*,\sigma[}$ and $\bar{Y}$ can only jump downwards at σ. Indeed, on the set B_n, by means of (3.15), (3.13) and (3.14) holds

$$\begin{aligned}\bar{Y}_{\sigma_n -} &= \mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{\varepsilon}{2} = \mathcal{E}^{g,+}_{\tau^*}(\xi) - \mathcal{E}^{g}_{\sigma_n}(\xi) + \mathcal{E}^{g}_{\sigma_n}(\xi) + \frac{\varepsilon}{2} \\ &\geq -\frac{\varepsilon}{8} + \mathcal{E}^{g}_{\sigma_n}(\xi) + \frac{\varepsilon}{2} \geq \tilde{Y}^n_{\sigma_n} - \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \tilde{Y}^n_{\sigma_n} = \bar{Y}_{\sigma_n}\,.\end{aligned}$$

Consequently,

$$\begin{aligned}&\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u \\ &\qquad \geq \bar{Y}_{(\sigma\vee s)\wedge t} - \int_{(\sigma\vee s)\wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma\vee s)\wedge t}^t \bar{Z}_u dW_u \geq \bar{Y}_t \,, \end{aligned} \quad (3.22)$$

where the second inequality in (3.22) follows from $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ and Proposition 3.2.

Step 5: The maximal element (Y^M, Z^M). By Zorn's lemma, there exists a maximal element (Y^M, Z^M) in $\mathcal{A}(\xi, g)$ with respect to the preorder (3.8), satisfying, without loss of generality, $Y^M_T = \xi$. By showing that the corresponding stopping time satisfies $\tau^M = T$, we have obtained a supersolution (Y^M, Z^M) satisfying $\|Y^M - \mathcal{E}^{g,+}(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon$, due to the definition of τ^M in analogy to (3.9). Thus, choosing $Y^M = Y^\varepsilon$ in (3.7) would finish our proof.
But on $\{\tau^M < T\}$ we could consider the chain consisting only of (Y^M, Z^M) and, analogously to (3.15) and (3.16), construct an upper bound $(\bar{Y}, \bar{Z})$, with corresponding stopping time $\tau(\bar{Y})$ as in (3.9), satisfying $P[\tau(\bar{Y}) > \tau^M \,|\, \tau^M < T] = 1$. This yields $P[\tau^M < T] \leq P[\tau(\bar{Y}) > \tau^M] = 0$, due to the maximality of τ^M. Hence we deduce that $\tau^M = T$. □

The techniques used in the proof of Theorem 3.4 show that $\mathcal{A}(\xi, g)$ exhibits a certain closedness under monotone limits of decreasing supersolutions.

Theorem 3.5. *Let g be a generator satisfying* (POS) *and* (NOR) *and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. Let furthermore $((Y^n, Z^n))$ be a decreasing sequence within $\mathcal{A}(\xi, g)$ with pointwise limit $\bar{Y}_t := \lim_n Y^n_t$, for $t \in [0, T]$. Then $\bar{Y}$ is a supermartingale and it holds*

$$\bar{Y}_t \geq \hat{Y}_t := \lim_{\substack{s\downarrow t \\ s\in\mathbb{Q}}} \bar{Y}_s \quad \textit{for all } t \in [0, T)\,. \quad (3.23)$$

Moreover, with $\hat{Y}_T := \xi$*, there is a sequence* $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ *such that* $\lim_n \|\tilde{Y}^n - \hat{Y}\|_{\mathcal{R}^\infty} = 0$*, and a unique control* $\hat{Z} \in \mathcal{L}$ *such that* $(\hat{Y}, \hat{Z}) \in \mathcal{A}(\xi, g)$.

Proof. First, $\bar{Y}$ is a supermartingale by monotone convergence. Inequality (3.23) is then proved analogously to (3.5) as in [6, Proposition 3.5]. The rest follows by adapting all steps in the proof of Theorem 3.4 and replacing $\mathcal{E}^{g,+}(\xi)$ by $\hat{Y}$. □

In a next step, we turn our focus to the question whether it is possible to find a minimal supersolution within $\mathcal{A}(\xi, g)$, the associated control process Z of which belongs to $\mathcal{L}^1$ and therefore $\int Z dW$ constitutes a true martingale instead of only a supermartingale. To this end we consider the following subset of $\mathcal{A}(\xi, g)$

$$\mathcal{A}^1(\xi, g) := \left\{ (Y, Z) \in \mathcal{A}(\xi, g) \, : \, Z \in \mathcal{L}^1 \right\} . \tag{3.24}$$

By imposing stronger assumptions on the terminal condition ξ, the next theorem yields the existence of a unique minimal supersolution in $\mathcal{A}^1(\xi, g)$.

Theorem 3.6. *Assume that the generator* g *satisfies* (POS) *and* (NOR) *and let* $\xi \in L^0(\mathcal{F}_T)$ *be a terminal condition such that* $(E[\xi^- \mid \mathcal{F}_\cdot])^* \in L^1(\mathcal{F}_T)$*. If* $\mathcal{A}^1(\xi, g) \neq \emptyset$*, then there exists a unique* $\hat{Z}$ *such that* $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^1(\xi, g)$.

Proof. $\mathcal{A}^1(\xi, g) \neq \emptyset$ yields that $\mathcal{A}(\xi, g) \neq \emptyset$, because $\mathcal{A}^1(\xi, g) \subseteq \mathcal{A}(\xi, g)$. Also, from $(E\left[\xi^- \mid \mathcal{F}_\cdot\right])^*_T \in L^1(\mathcal{F}_T)$ we deduce that $\xi^- \in L^1(\mathcal{F}_T)$. Hence, Theorem 3.4 yields the existence of an unique control $\hat{Z}$, such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. Verifying that $\hat{Z} \in \mathcal{L}^1$ is done as in [6, Theorem 4.5]. □

3.3 Relaxations of the Conditions (NOR) and (POS)

In this section we discuss possible relaxations of the conditions (NOR) and (POS) imposed on the generator throughout Sections 3.1 and 3.2.
First, we want to replace (NOR) by the weaker assumption (NOR'). We say that a generator g satisfies

(NOR') if, for all $\tau \in \mathcal{T}$, there exists some stopping time $\delta > \tau$ such that the stochastic differential equation

$$dy_s = g_s(y_s, 0) ds \, , \quad y_\tau = \mathcal{E}^{g,+}_\tau(\xi) + \frac{\varepsilon}{2} \tag{3.25}$$

admits a solution on $[\tau, \delta]$.

By this we obtain the following corollary to Theorem 3.4.

Corollary 3.7. *Let* g *be a generator satisfying* (POS) *and* (NOR') *and* $\xi \in L^0(\mathcal{F}_T)$ *a terminal condition such that* $\xi^- \in L^1(\mathcal{F}_T)$*. If* $\mathcal{A}(\xi, g) \neq \emptyset$*, then there exists a unique* $\hat{Z} \in \mathcal{L}$ *such that* $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.

Proof. We will proceed along the lines of the proof of Theorem 3.4 with the focus on the required alterations.
The first difficulty lies in the pasting at the stopping time τ^* within the definition of $(\bar{Y}, \bar{Z})$ in (3.15) and (3.16). Instead of extending by a constant function, we concatenate the value process at τ^* with the solution of the SDE (3.25), started at $y_{\tau^*} = \mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{\varepsilon}{2}$ and denoted by y. We emphasize that the zero control is maintained.

Furthermore, we have to introduce an additional stopping time and adjust $\bar{\tau}$ defined in (3.11), in order to ensure that our constructed value process does not leave the ε-neighborhood of $\mathcal{E}^{g,+}(\xi)$. We define

$$\kappa := \inf\{t > \tau^* \,:\, 1_{\{\tau^*<T\}} \int_{\tau^*}^{t} g_s(y_s, 0)ds > \frac{\varepsilon}{6}\} \wedge \delta \tag{3.26}$$

$$\text{and} \quad \bar{\tau} := \inf\{t > \tau^* \,:\, 1_{\{\tau^*<T\}} |\mathcal{E}^{g,+}_{\tau^*}(\xi) - \mathcal{E}^{g,+}_t(\xi)| > \frac{\varepsilon}{6}\} \wedge T \tag{3.27}$$

and use $\bar{\kappa} := \kappa \wedge \bar{\tau}$ within the definition of the sequence (σ_n) in analogy to (3.12), that is, $\sigma_n = \hat{\sigma}_n \wedge \bar{\kappa}$, for all $n \in \mathbb{N}$. As before, we set $\sigma := \sum_{n\geq 1} 1_{B_n}\sigma_n$.
The pasting in (3.15) and (3.16) is done analogously to the proof of Theorem 3.4, but now with the distinction that $\bar{Y}1_{[\tau^*,\sigma[} = y_{\tau^*} + 1_{[\tau^*,\sigma[} \int_{\tau^*}^{\cdot} g_s(y_s, 0)ds$. The definition of the stopping times κ, $\bar{\tau}$ and σ implies that, on the set B_n, we have $\bar{Y}_t \leq \mathcal{E}^{g,+}_t(\xi) + \varepsilon$, for all $t \in [\tau^*, \sigma_n[$. Indeed, observe that, for $t \in [\tau^*, \sigma_n[$,

$$\begin{aligned}\bar{Y}_t = \mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{\varepsilon}{2} + \int_{\tau*}^{t} g_s(y_s, 0)ds \leq \mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{2\varepsilon}{3} \\ = \mathcal{E}^{g,+}_{\tau^*}(\xi) - \mathcal{E}^{g,+}_t(\xi) + \mathcal{E}^{g,+}_t(\xi) + \frac{2\varepsilon}{3} \leq \mathcal{E}^{g,+}_t(\xi) + \frac{\varepsilon}{6} + \frac{2\varepsilon}{3} < \mathcal{E}^{g,+}_t(\xi) + \varepsilon\,,\end{aligned}$$

by means of (3.26) and (3.27), together with the definition of σ_n. Furthermore, on the set B_n,

$$\bar{Y}_{\sigma_n-} = \mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{\varepsilon}{2} + \int_{\tau*}^{\sigma_n} g_s(y_s, 0)ds \geq \mathcal{E}^{g,+}_{\tau^*}(\xi) + \frac{\varepsilon}{2} \geq \bar{Y}_{\sigma_n}\,. \tag{3.28}$$

The first inequality in (3.28) follows from (POS), whereas the second is proved analogously to the proof of Theorem 3.4, using (3.27) and the definition of σ_n. Hence, pasting at the stopping time σ is in accordance with Proposition 3.2.
Finally, the downward jumps at τ^* and at σ, together with the zero control in between, ensure that $(\bar{Y}, \bar{Z})$ satisfies (3.1), as was shown in Step 4c of the proof of Theorem 3.4. The rest of the proof does not need any further alterations. □

Also the positivity assumption (POS) on the generator can be relaxed to a linear bound below, which however has to be consistent with the assumption (NOR'). In the following we say that a generator g is

(LB-NOR') linearly bounded from below under (NOR'), if there exist adapted measurable processes a and b with values in $\mathbb{R}^{1\times d}$ and $\mathbb{R}$, respectively, such that $g(y, z) \geq az^T - b$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1\times d}$, and

$$\frac{dP^a}{dP} = \mathcal{E}\left(\int adW\right)_T \tag{3.29}$$

defines an equivalent probability measure P^a. Furthermore, $\int_0^t b_s ds \in L^1(P^a)$ holds for all $t \in [0, T]$, and a and b are such that the positive generator defined by

$$\bar{g}(y, z) := g\left(y + \int_0^{\cdot} b_s ds, z\right) - az^T - b\,, \quad \text{for all } (y, z) \in \mathbb{R} \times \mathbb{R}^{1\times d}\,, \tag{3.30}$$

satisfies (NOR').

An (LB-NOR') setting can always be reduced to a setting with generator satisfying (POS) and (NOR'), by using the change of measure (3.29) and $\bar{g}$ defined in (3.30). Hence, Lemma 3.1 and Proposition 3.2, which strongly rely on the property (POS), can be applied. Note that for the case $b = 0$, the generator $\bar{g}$ even satisfies (POS) and (NOR). However, we need a slightly different definition of admissibility than before. A control process Z is said to be *a-admissible*, if $\int Z dW^a$ is a P^a-supermartingale, where $W^a = \left(W^1 - \int a^1 ds, \cdots, W^d - \int a^d ds\right)^T$ is a P^a-Brownian motion by Girsanov's theorem.

The set $\mathcal{A}^a(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} \, : \, Z$ is a-admissible and (3.1) holds$\}$, as well as the process

$$\mathcal{E}_t^{g,a}(\xi) = \operatorname{ess\,inf}\{Y_t \in L^0(\mathcal{F}_t) \, : \, (Y, Z) \in \mathcal{A}^a(\xi, g)\}\,, \quad \text{for } t \in [0, T]\,,$$

are defined analogously to (3.2) and (3.3), respectively. We are now ready to state our most general result, which follows from Corollary 3.7 and [6, Theorem 4.16].

Theorem 3.8. *Let g be a generator satisfying* (LB-NOR') *and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(P^a)$. If in addition $\mathcal{A}^a(\xi, g) \neq \emptyset$, then there exists a unique a-admissible control $\hat{Z}$ such that $(\mathcal{E}^{g,a}(\xi), \hat{Z}) \in \mathcal{A}^a(\xi, g)$.*

3.4 Continuous Local Martingales and Controls in $\mathcal{L}^1$

Under stronger integrability conditions, the techniques used in the proof of Theorem 3.4 can be generalized to the case where the Brownian motion W appearing in the stochastic integral in (3.1) is replaced by a d-dimensional continuous local martingale M. Let us assume that M is adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, which satisfies the usual conditions and in which all martingales are continuous and all stopping times are predictable. We consider controls within the set $\mathcal{L}^1 := \mathcal{L}^1(M)$, consisting of all $\mathbb{R}^{1\times d}$-valued, progressively measurable processes Z, such that $\int Z dM \in \mathcal{H}^1$. As before, for $Z \in \mathcal{L}^1$ the stochastic integral $(\int_0^t Z_s dM_s)_{t\in[0,T]}$ is well defined and is by means of the Burkholder-Davis-Gundy inequality a continuous martingale. A pair $(Y, Z) \in \mathcal{S} \times \mathcal{L}^1$ is now called a supersolution of a BSDE, if it satisfies, for $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u) d\left\langle M\right\rangle_u + \int_s^t Z_u dM_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi\ , \tag{3.31}$$

for a normal integrand g as generator and a terminal condition $\xi \in L^0(\mathcal{F}_T)$. We will focus on the set

$$\mathcal{A}^{M,1}(\xi, g) := \left\{(Y, Z) \in \mathcal{S} \times \mathcal{L}^1 \ : \ (Y, Z) \text{ satisfy (3.31)}\right\}\ .$$

If we assume $\mathcal{A}^{M,1}(\xi, g)$ to be non-empty, Theorem 3.4 combined with compactness results for sequences of $\mathcal{H}^1$-bounded martingales given in Delbaen and Schachermayer [3] yields that

$$\mathcal{E}_t^g(\xi) := \operatorname{ess\,inf}\left\{Y_t \in L^0(\mathcal{F}_t) \, : \, (Y, Z) \in \mathcal{A}^{M,1}(\xi, g)\right\}\ , \quad t \in [0, T]\,, \tag{3.32}$$

is the value process of the unique minimal supersolution within $\mathcal{A}^{M,1}(\xi, g)$. Note that Lemma 3.1 and Proposition 3.2 extend to the case where W is substituted by M.

Theorem 3.9. *Assume that the generator g satisfies* (POS) *and* (NOR) *and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $(E[\xi^- \,|\, \mathcal{F}_\cdot])^* \in L^1(\mathcal{F}_T)$. If $\mathcal{A}^{M,1}(\xi, g) \neq \emptyset$, then there exists a unique $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^{M,1}(\xi, g)$.*

Proof. By assumption, there is some $(Y^b, Z^b) \in \mathcal{A}^{M,1}(\xi, g)$ and we consider, without loss of generality, only those pairs $(Y, Z) \in \mathcal{A}^{M,1}(\xi, g)$ satisfying $Y \le Y^b$, obtained by suitable pasting as in Proposition 3.2. Using the techniques of the proof of Theorem 3.4, we can find a sequence $((Y^n, Z^n)) \subset \mathcal{A}^{M,1}(\xi, g)$ satisfying $\lim_n \|Y^n - \mathcal{E}^{g,+}(\xi)\|_{\mathcal{R}^\infty} = 0$, in analogy to (3.6). Since $(\int Z^n dM)$ is uniformly bounded in $\mathcal{H}^1$, compare [6, Theorem 4.5], it follows from [1, Theorem 1] that $\mathcal{E}^{g,+}(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^{g,+}(\xi) = \mathcal{E}_0^{g,+}(\xi) + N - A$ and that

$$\lim_{n\to\infty} \left\| \int Z^n dM - N \right\|_{\mathcal{H}^1} = 0\,. \tag{3.33}$$

Moreover, $N \in \mathcal{H}^1$. Now [3, Theorem 1.6] yields the existence of some $\hat{Z} \in \mathcal{L}^1$ such that $N = \int \hat{Z} dM$. By means of (3.33), (Z^n) converges, up to a subsequence, $P \otimes d\left\langle M \right\rangle_t$-almost surely to $\hat{Z}$ and $\lim_n \int_0^t Z^n dM = \int_0^t \hat{Z} dM$, for all $t \in [0, T]$, P-almost surely, by means of the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n\to\infty} Z^n(\omega) = \hat{Z}(\omega)$, $d\left\langle M \right\rangle_t$-almost surely, for almost all $\omega \in \Omega$. Verifying that $(\mathcal{E}^{g,+}(\xi), \hat{Z})$ satisfy (3.31) is now done analogously to Step 1 in the proof of Theorem 3.4, and hence we are done. $\square$

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